Sturm-Liouville Theory

1-D Heat eq: $u_t = k u_{xx}$ $\quad 0 < x < L$

we've seen that if $u(0,t) = u(L,t) = 0$ (ends frozen)

eigenvalues: $\lambda_n = \dfrac{n^2\pi^2}{L^2}$ eigenfunctions $X_n = \sin\left(\dfrac{n\pi}{L}x\right)$ $n = 1, 2, 3, \ldots$

and if $u_x(0,t) = u_x(L,t) = 0$ (ends insulated)

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad X_n = \cos\left(\frac{n\pi}{L}x\right) \qquad n = 0, 1, 2, 3, \ldots$$

in both cases, the eigenvalues are the frequencies of each mode of the solutions $\rightarrow$ integer multiples of $\frac{\pi}{L}$

also, the eigenfunctions are mutually orthogonal: $\int_0^L X_n X_m \, dx = 0$ if $n \neq m$

will these still be true if we used more complicated boundary conditions?

for example, $u_t = k u_{xx}$ $\quad 0 < x < L$

$u(0,t) = 0$ left end frozen

$\underbrace{u_x(L,t) = -h\,u(L,t)}$ $\quad h > 0$

this models a heat exchange at $x = L$

(from Fourier's law and Newton's law of cooling)

after separation of variables, we get

$$X'' + \lambda X = 0$$

$$X(0) = 0$$

$$X'(L) + h X(L) = 0$$

using $X(0) = 0$, we find $X = B\sin(\sqrt{\lambda}\,x)$ $\longrightarrow$ eigenfunctions are $X_n = \sin(\sqrt{\lambda_n}\,x)$

$$X' = \sqrt{\lambda}\,B\cos(\sqrt{\lambda}\,x)$$

$$X'(L) + hX(L) = 0 \to \sqrt{\lambda}\,B\cos(\sqrt{\lambda}\,L) + Bh\sin(\sqrt{\lambda}\,L) = 0$$

we require $\lambda \neq 0$, $B \neq 0$

$$\sqrt{\lambda}\cos(\sqrt{\lambda}L) = -h\sin(\sqrt{\lambda}L)$$

$$\boxed{\tan(\sqrt{\lambda}L) = -\frac{\sqrt{\lambda}}{h}}$$ solve for λ

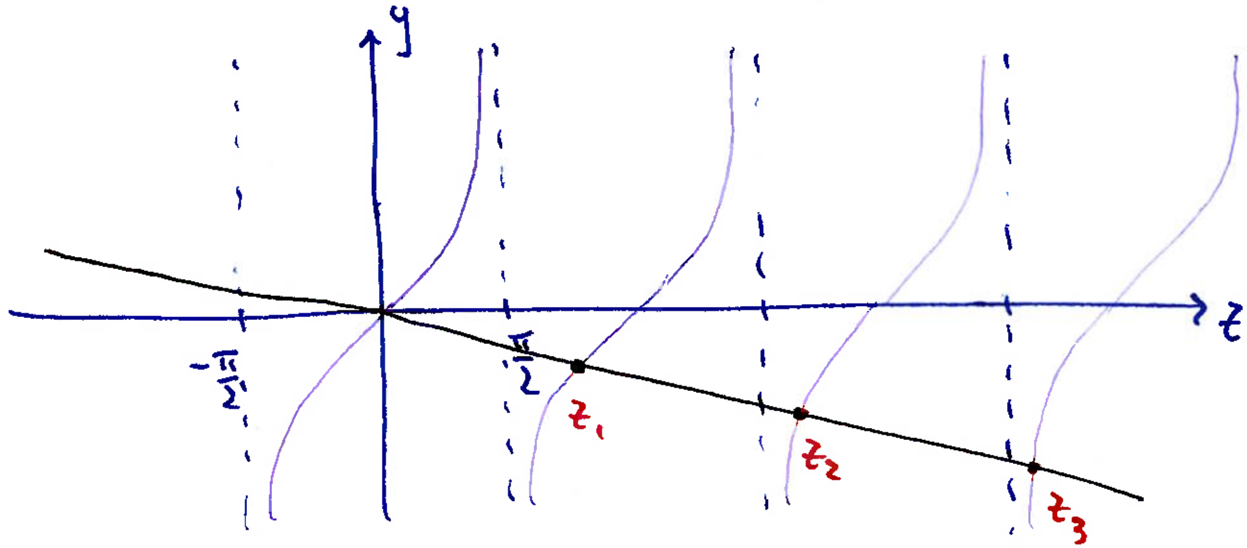
(transcendental eq: variable we want is on both sides)

let's try to interpret the solution graphically

let's define $z = \sqrt{\lambda}L$ $(z > 0)$

then $\tan(z) = -\frac{z}{hL}$ is describing the intersections

of $y = \tan(z)$ and $y = -\frac{1}{hL}z$



there are infinitely-many intersections $z_1, z_2, z_3, \ldots$

whatever they are, $z = \sqrt{\lambda}L$ no longer leads to

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

but notice as $n \to \infty$, z_n gets closer to multiples of $\frac{\pi}{2}$

(left asymptote of each cycle of tangent)

→ eventually start to resemble the λ of the two basic cases

what if the diffusivity is not constant?

$u_t = k(x)\,u_{xx}$ what would X_n look like? λ_n ?

collectively, this is what the Sturm-Liouville theory can answer

Sturm-Liouville Problem

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda w(x)y = 0 \qquad a < x < b$$

Subject to

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0 \qquad \alpha_1, \alpha_2 \text{ not both zero}$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0 \qquad \beta_1, \beta_2 \text{ not both zero}$$

notice if $p=1$, $q=0$, $w=1$ we get $y'' + \lambda y = 0$ $(X'' + \lambda X = 0)$

Solutions are Fourier series

$p = x$, $q = -\frac{n^2}{x}$, $w = x$ we get $xy'' + y' + (\lambda x - \frac{n^2}{x})y = 0$

(models waves of a circular drum)

Solutions are Bessel functions

$p = 1-x^2$, $q=0$, $w=1$ we get $(1-x^2)y'' - 2xy' + \lambda y = 0$

(models the steady state solution
of a heated sphere)

Solutions are Legendre polynomials

big picture view: choosing p, q, w can give us a wide
variety of heat / wave situations